Inversion of conductivity profiles using the Volterra functional method

Lu-Jun Chen,[1] Cuen-Zhi Li,[1] and Chang-Hong Liang[2]
[1]*Physics Department, Box 271, Xidian University, Xi'an, 710071, People's Republic of China*
[2]*Electromagnetic Field Engineering Department, Xidian University, Xi'an, 710071, People's Republic of China*
(Received 8 December 1993; revised manuscript received 24 March 1994)

The Volterra functional method is used for the electromagnetic inverse problem. A general analytical procedure and symbolic computer code implementation are successfully constructed that provides an alternative way for the inverse problem. The conductivity profiles, as an example, are reconstructed by the method. The results show that using only three terms of an expansion gives an obvious improvement when compared with former approximations.

PACS number(s): 03.50.De, 84.40.Cb

I. INTRODUCTION

There have been many interesting results in the reconstruction of electromagnetic parameters of one-dimensional distribution, such as the direct formula for dielectric medium profiles given by Ladouceur and Jordan [1,2]. Recently, Ge and Chen [3] (one of the authors of this paper) derived an explicit formula for the conductivity profile of the conducting medium. Another result was obtained by Cui and Liang [4] by using the microwave networking technique. In this paper, we present a very different method for reconstructing the conductivity profile: the Volterra functional method. Beginning with the Riccati equation, a general analytical procedure and symbolic code implementation are successfully constructed, and a result with three terms expansion is obtained. Some numerical experiments show that the results given by the method of this paper give better results in comparison with former approximations.

II. ANALYTICAL PROCEDURE AND FORMULA

Let us consider a half-space problem in one dimension with an interface at $z=0$. The medium of interest occurs in the right region $z>0$. A plane wave, of which the electric field **E** is perpendicular to the plane of incidence, impinges on the interface from the left region $z<0$. In the right region $z>0$, by introducing a wave impedance [5]

$$Z(z,k)=\frac{1}{\xi_0}\frac{E_x}{H_y}\ , \tag{1}$$

and, according to electromagnetic theory [6], one obtains

$$\frac{dZ}{dz}=jk_0\left[\left[\cos^2\theta-j\frac{\xi_0\sigma}{k_0}\right]Z^2-1\right]\ , \tag{2}$$

where k and ξ_0 are the wave number and wave impedance in free space, and θ is the angle of incidence. When $\theta=0$, Eq. (2) becomes

$$\frac{dZ}{dz}=jk_0\left[\left[1-j\frac{\xi_0\sigma}{k_0}\right]Z^2-1\right]\ . \tag{3}$$

A relation between reflection coefficient $r(k)$ and wave impedance $Z(z,k)$,

$$r(k)=\frac{Z(0,k)-1}{Z(0,k)+1} \tag{4}$$

is obtained according to Eq. (1). Here (and below), k_0 is simply denoted by k. In free space, it can be proved that $Z(z,k)=1$. In the medium, $Z(z,k)$ is expanded as

$$Z(z,k)=1+\epsilon Z_1(z,k)+\epsilon^2 Z_2(z,k)+\cdots\ , \tag{5}$$

where ϵ is a dummy variable (used to keep track of a successive order, whose value is finally set equal to 1). The above expansion is valid only for the case when the reflection coefficient is not too large. Since the reflection information is determined by the distribution of the $\sigma(z)$ profiles, $r(k)$ and each $Z_p(z,k)$ $(p=1,2,\ldots)$ can be regarded as functionals of $\sigma(z)$. The functional for $r(k)$ can be expanded by multilinear operators, i.e.,

$$r(k)=\tfrac{1}{2}\sum_{p=0}^{\infty}\epsilon^p\hat{A}_p[\sigma,\sigma,\ldots,\sigma]\ , \tag{6}$$

where $\hat{A}_p$ is a multilinear operator and has a linear dependence on each one of its p arguments [i.e., $\sigma(z)$ appears p times in $\tilde{A}_p$]. On the other hand, substituting Eq. (5) into Eq. (4) gives

$$\begin{aligned}r(k)=&\tfrac{1}{2}Z_1(0,k)\epsilon+\tfrac{1}{4}[2Z_2(0,k)-Z_1^2(0,k)]\epsilon^2\\&+\tfrac{1}{2}[Z_3(0,k)-Z_1(0,k)Z_2(0,k)\\&+\tfrac{1}{4}Z_1^3(0,k)]\epsilon^3+\cdots\ .\end{aligned} \tag{7}$$

Comparing Eqs. (6) and (7), one obtains

$$\begin{aligned}&\hat{A}_1[\sigma(z)]=Z_1(0,k)\ ,\\&\hat{A}_2[\sigma(z),\sigma(z)]=\tfrac{1}{2}[2Z_2(0,k)-Z_1^2(0,k)]\ ,\\&\hat{A}_3[\sigma(z),\sigma(z),\sigma(z)]\\&\qquad=[Z_3(0,k)-Z_1(0,k)Z_2(0,k)+\tfrac{1}{4}Z_1^3(0,k)]\ ,\\&\vdots\ .\end{aligned} \tag{8}$$

The above equations determine the connotations of each multilinear operator $\hat{A}_p$. Below, the $\sigma(z)$ dependence of $Z_p(0,k)$ will be carried out. Substituting Eq. (5) into the Riccati equation, i.e., Eq. (3), gives

$$\frac{dZ_p}{dz}=j2kZ_p+F_p[Z_1,Z_2,\ldots,Z_{p-1};\sigma(z)] \quad (p=1,2,\ldots)\ , \tag{9}$$

where

$$\begin{aligned} F_1&=\xi_0\sigma(z)\sim O(\epsilon^1)k\ ,\\ F_2&=2\xi_0\sigma(z)Z_1+jkZ_1^2\ ,\\ F_3&=\xi_0\sigma(z)[Z_1^2+2Z_2]+j2kZ_1Z_2\ ,\\ &\vdots \end{aligned} \tag{10}$$

and Z_p simply means

$$Z_p(z,k)=Z_p[z,k;\sigma(z),\sigma(z),\ldots,\sigma(z)]\ ,$$

where $\sigma(z)$ appears p times.

Solving differential equations Eq. (9), we have

$$Z_p(z,k)=Z_p(0,k)e^{j2kz}+e^{j2kz}\int_0^x F_p[Z_1(z',k),Z_2(z',k),\ldots,Z_{p-1}(z',k);\sigma(z')]e^{-j2kz'}dz'\ , \tag{11}$$

where $Z_p(0,k)$ can be obtained by making a Fourier transformation of Eq. (9), i.e.,

$$Z_p[0,k;\sigma(z),\ldots,\sigma(z)]=-\int_0^{+\infty}F_p[Z_1(z',k),Z_2(z',k),\ldots,Z_{p-1}(z',k);\sigma(z')]e^{-j2kz'}dz'\ . \tag{12}$$

It is clear that term $Z_p(z,k)$ is determined by making use of Eqs. (10), (11), and (12). By using Eq. (12), Eq. (11) can be simplified as

$$Z_p[z,k;\sigma,\sigma,\ldots,\sigma]=-e^{j2kz}\int_z^{+\infty}F_p[Z_1(z',k),\ldots,Z_{p-1}(z',k);\sigma(z')]e^{-j2kz'}dz'\ . \tag{13}$$

Equations (8), (12), and (13) give just the concrete meanings of each operator $\hat{A}_p$. Substituting F_1 in Eq. (10) into Eqs. (12) and (13), $Z_1(0,k)$ and $Z_1(z,k)$ can be calculated. Substituting $Z_1(0,k)$, $Z_1(z,k)$, and F_2 in Eq. (10) into Eqs. (12) and (13), $Z_2(0,k)$ and $Z_2(z,k)$ can be calculated, and so on. As an example, the final results of each $Z_p(0,k)$ in $\hat{A}_2$, $\hat{A}_3$ and $\hat{A}_3$, respectively, are of the forms,

$$Z_1[0,k;\sigma]=-\xi_0\int_0^{+\infty}\sigma(z)e^{-j2kz}dz\ , \tag{14}$$

$$Z_2[0,k;\sigma,\sigma]=\tfrac{1}{2}Z_1^2(0,k)-\xi_0^2\int_0^{+\infty}dz\left[\sigma(z)\int_z^{+\infty}\sigma(z')e^{-j2kz'}dz'\right]\ , \tag{15}$$

$$\begin{aligned} Z_3[0,k;\sigma,\sigma,\sigma]=&Z_1(0,k)Z_2(0,k)-\tfrac{1}{4}Z_1^3(0,k)-\tfrac{1}{4}\xi_0^3\int_0^{+\infty}\left[\int_z^{+\infty}\sigma(z')e^{-jk2kz'}dz'\right]^2e^{j2kz}\sigma(z)dz\\ &-\xi_0^3\int_0^{+\infty}\left[\int_z^{+\infty}\sigma(z')\left[\int_{z'}^{+\infty}\sigma(z'')e^{-j2kz''}dz''\right]dz'\right]\sigma(z)dz\ ,\\ &\vdots\ . \end{aligned} \tag{16}$$

Making an inverse Fourier transformation of Eq. (14), one obtains

$$\sigma(z)=-\frac{2\epsilon_0}{\pi}\left[\int_0^{+\infty}Z_1(0,k)e^{j\omega t}d\omega\right]\Bigg|_{t=2z/c}\ . \tag{17}$$

Note that $Z_1(0,k)$ is the first term of Eq. (7), if $Z_1(0,k)$ in Eq. (17) is substituted as an approximation by $2r(k)$, and the result of $\sigma(z)$ in Eq. (17) obtained by such a substitution is symbolized by $\sigma_1(z)$, so that one obtains

$$\sigma_1(z)=-4\epsilon_0R(t)|_{t=2x/c}\ , \tag{18}$$

where

$$R(t)=\frac{1}{\pi}\int_0^{+\infty}r(k)e^{j\omega t}d\omega \tag{19}$$

is the reflection coefficient in the time domain. Equation (18) will be taken as the expansion center of the Volterra functional series for conductivity profiles, i.e.,

$$\begin{aligned} \sigma(z)&=\sum_{\pi=1}^{\infty}\sigma_n(z)\ ,\\ \sigma_n(z)&=\int_0^{+\infty}\int_0^{+\infty}\cdots\int_0^{+\infty}\sigma_1(x_1)\sigma_1(x_2)\ldots\sigma_1(x_n)\\ &\quad\times\gamma^{(n)}(z;x_1,x_2,\ldots,x_n)\\ &\quad\times dx_1dx_2\ldots dx_n\ , \end{aligned} \tag{20}$$

which is the generalization of Taylor series in functional. It can clearly be seen that the first Volterra kernel $\gamma^{(1)}$ is

$$\gamma^{(1)}(z;x_1)=\delta(x_1-z)\ .$$

The other kernels $\gamma^{(n)}$ will be determined in the following. Substituting Eq. (20) into Eq. (6) and considering the linearity of operators $\hat{A}_p$ on each argument, one can obtain

$$\hat{A}_1[\sigma_1]=2r(k)\ , \tag{21}$$

$$\hat{A}_1[\sigma_2]=-\hat{A}_2[\sigma_1,\sigma_1]\ , \tag{22}$$

$$\hat{A}_1[\sigma_3]=-\hat{A}_2[\sigma_1,\sigma_2]-\hat{A}_2[\sigma_2,\sigma_1]-\hat{A}_3[\sigma_1,\sigma_1,\sigma_1]\ ,$$
$$\vdots \qquad (23)$$

where

$$\hat{A}_2[\sigma_1,\sigma_2]=Z_2(0,k;\sigma_1,\sigma_2)-\tfrac{1}{2}Z_1(0,k;\sigma_1)Z_1(0,k;\sigma_2)\ ,$$
$$\hat{A}_2[\sigma_2,\sigma_1]=Z_1(0,k;\sigma_2,\sigma_1)-\tfrac{1}{2}Z_1(0,k;\sigma_2)Z_1(0,k;\sigma_1)\ ,$$
$$\vdots\ .$$

Comparing Eq. (21) with Eqs. (18) and (17) gives

$$\hat{A}_1^{-1}(\ )=-\frac{2\epsilon_0}{\pi}\int_0^{+\infty}(\ )e^{j\omega t}d\omega\ .$$

By using Eqs. (8), (14), and (15), Eq. (22) gives

$$\sigma_2(z)=-\hat{A}_1^{-1}\{\hat{A}_2[\sigma_1,\sigma_1]\}=\zeta_0\sigma_1(z)\int_0^x\sigma_1(z')dz'\ , \qquad (24)$$

which also means

$$\gamma^{(2)}(z;x_1,x_2)=\zeta_0\delta(x_1-z)H(z-x_2)\ ,$$

where $H(z-x_2)$ is the Heaviside function. The same way, one obtains

$$\sigma_3(z)=-\hat{A}_1^{-1}\{\hat{A}_2[\sigma_1,\sigma_2]+\hat{A}_2[\sigma_2,\sigma_1]+\hat{A}_3[\sigma_1,\sigma_1,\sigma_1]\}$$
$$=2\zeta_0^2\left[\sigma_1(z)\int_0^z\left[\int_0^x\sigma_1(x')dx'\right]\sigma_1(x)dx-\frac{1}{8}\int_0^z\left[\int_x^z\sigma_1(x')\sigma_1(z+x-x')dx'\right]\sigma_1(x)dx\right]\ , \qquad (25)$$

which also gives the concrete meaning of the third hernal

$$\gamma^{(3)}(z;x_1,x_2,x_3)=2\zeta_0^2\delta(x_1-z)H(z-x_2)H(x_3-x_2)$$
$$-\tfrac{1}{4}\zeta_0^2H(z-x_1)[H(z-x_2)-H(x_1-x_2)]\delta(x_3-z-x_1+x_2)\ .$$

By using Eq. (18), the functional expansion of $\sigma(z)$ up to three terms can be written as

$$\sigma(z)=-4\epsilon_0\left[R(t)\left[1-2\int_0^tR(t')dt'+8\int_0^tdt'R(t')\int_0^{t'}R(t'')dt''\right]\right.$$
$$\left.\left.-\int_0^tdt'R(t')\int_{t'}^tR(t'')R(t+t'-t'')dt''\right]\right|_{t=2z/c}\ . \qquad (26)$$

For the case $\theta\neq 0°$, by making the transformations $k\to k'=k\cos\theta$ (equivalent to $z\to z'=z\cos\theta$), $Z\to Z'=Z\cos\theta$ (equivalent to $R\to R'(t)=R(t)\cos\theta$), $\sigma\to\sigma'=\sigma/\cos\theta$, Eq. (2) is transformed to be Eq. (3). The final results of Eq. (26) after the transformation are to make the changes

$$-4\epsilon_0\to-4\epsilon_0\cos^2\theta,\quad t=2z/c\to t=2z\cos\theta/c\ .$$

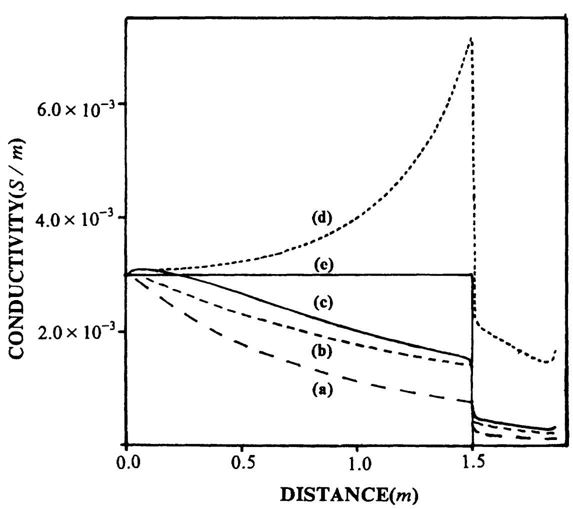

FIG. 1. Comparison of three reconstruction results for a single slab medium. (a) The Born approximation, which is just the first term of this paper's results. (b) The results form Ref. [3]. (c) The results of this paper by three functional terms, i.e., by Eq. (26), where $\sigma=0.003$ corresponds to $\xi_0\sigma=1.13$ with $\xi_0=120\pi\Omega$. (d) The results from Ref. [4]. (e) The ideal profiles. Note that the ordinate is scaled by $\xi_0\sigma(z)$ (within the region $\xi_0\sigma(z)\leq 1$) in Ref. [4], not by $\sigma(z)$ as in this paper.

III. RESULTS ANALYSIS

To test the applicability of this scheme, two examples with one and two piecewise homogeneous stratified media are considered. Based on Eq. (26), the reconstruction of

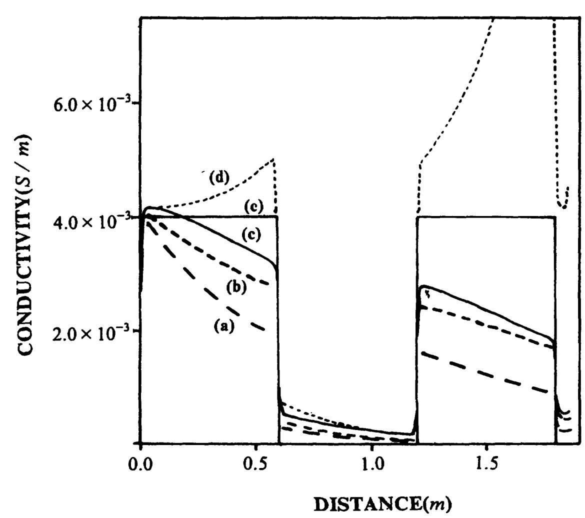

FIG. 2. Comparison of three reconstruction results for two slab media. (a), (b), (c), (d), and (e) denote the same meanings as in Fig. 1, where $\sigma=0.004$ corresponds to $\xi_0\sigma=1.51$.

the conductivity profile from the reflection data can be carried out. The reflection coefficient $r(k)$, instead of experimental data, can be produced numerically according to electromagnetic theory. Then $R(t)$ can be calculated through the fast Fourier transform algorithm by Eq. (19). For comparing with the other results, five curves are given in Figs. 1 and 2, in which it is can be shown that a better reconstruction is obtained by using the method of this paper. However, there still is some discrepancy between the ideal profile and the profiles obtained by using Eq. (26). However, it would be expected that the results will be ameliorated by taking more terms into account, providing that, for practical problems, the information coming from the very deep layers (corresponding to the very lower frequency) is strong enough. Higher-order terms in the expansion may be beneficial to fit the curve into the ideal profiles. However, the terms higher than three order are difficult to reduce by handwork. However, by using a computer to reduce the procedure (such as REDUCE and MATHEMATICA), they are easily calculated according to the symbolic code and implementation scheme presented by this paper. Furthermore, this method can also be used in the reconstruction $\epsilon_r(z)-\sigma(z)$ profiles and $\epsilon_r(z)-\mu_r(z)$ profiles, and some results have been obtained which have taken in the continued composition. It should be pointed out that, for the practical problems, because the information coming from the deep layer (corresponding to the lower frequency) is very weak (this is the reason why a high-frequency approximation is reasonable). Therefore, even if a method which may effectually make use of all the frequency information including the very low-frequency signal were used, it would still be difficult to fit the reconstructed results into the practical profiles very well. Here we underline that the method of this paper provides a way for the reconstruction of conductivity profiles; in addition, it gives a better approximation than the former results [3,4].

[1] H. D. Ladouceur and A. K. Jordan, J. Opt. Soc. Am. A **2**, 1916 (1985).

[2] A. K. Jordan and H. D. Ladouceur, Phys. Rev. A **36**, 4245 (1987).

[3] D. B. Ge and L. J. Chen, IEEE Trans. Antennas Propagat. **39**, 907 (1991).

[4] T. J. Cui and C. H. Liang, Electron. Lett. **28**, 1349 (1992).

[5] G. Mazzarella and G. Panarielo, J. Opt. Soc. Am. A **8**, 1622 (1991).

[6] R. E. Collin, *Field Theory of Guided Waves* (McGraw-Hill, New York, 1960).